J. Math. Anal. Appl. 367 (2010) 146–153

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On nonstability of the linear recurrence of order one

Janusz Brzdęk [a], Dorian Popa [b,*], Bing Xu [c]

[a] Department of Mathematics, Pedagogical University, Podchorążych 2, PL-30-084 Kraków, Poland
[b] Technical University, Department of Mathematics, Str. C. Daicoviciu 15, Cluj-Napoca, 400020, Romania
[c] Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, PR China

ARTICLE INFO

Article history:
Received 15 October 2009
Available online 4 January 2010
Submitted by R. Curto

Keywords:
Hyers–Ulam stability
Nonstability
Linear recurrence
Shadowing
Banach space

ABSTRACT

We show that, under some assumptions, the linear recurrence (or difference equation) of order one in a Banach space is nonstable in the Hyers–Ulam sense. Our results are also connected with the notion of shadowing in dynamical systems and computer sciences.

In what follows $\mathbb{N}$ stands for the set of positive integers, $\mathbb{K}$ denotes either the field of reals $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$ and X is a Banach space over $\mathbb{K}$. Let $(\varepsilon_n)_{n\geqslant 0}$ be a sequence of positive real numbers, $(a_n)_{n\geqslant 0}$ a sequence in $\mathbb{K}\setminus\{0\}$ and $(b_n)_{n\geqslant 0}$ a sequence in X.

It is already known (see [14]; cf. also [3,4]) that, in the case where

$$\limsup_{n\to\infty}\frac{\varepsilon_n|a_{n+1}|}{\varepsilon_{n+1}}<1 \quad \text{or} \quad \liminf_{n\to\infty}\frac{\varepsilon_n|a_{n+1}|}{\varepsilon_{n+1}}>1, \tag{1}$$

for every sequence $(x_n)_{n\geqslant 0}$ in X satisfying the relation

$$\|x_{n+1}-a_nx_n-b_n\|\leqslant\varepsilon_n,\quad n\geqslant 0, \tag{2}$$

there exists a sequence $(y_n)_{n\geqslant 0}$ in X such that

$$y_{n+1}=a_ny_n+b_n,\quad n\geqslant 0, \tag{3}$$

and

$$L:=\sup_{n\in\mathbb{N}}\frac{\|x_n-y_n\|}{\varepsilon_{n-1}}<\infty, \tag{4}$$

which means that $\|x_n-y_n\|\leqslant L\varepsilon_{n-1}$ for $n\in\mathbb{N}$. In connection with this property there arises a natural question whether condition (1) can be weakened. Simple examples given in [14] show that if (1) does not hold, then analogous result is not generally true. A more precise and general statement concerning such a situation for the linear recurrence (of higher order and with constant coefficients) is obtained in [2].

* Corresponding author.
E-mail address: Popa.Dorian@math.utcluj.ro (D. Popa).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.12.052

The problem presented above is connected with the notion of the Hyers–Ulam stability of functional equations; for more information on this subject and some recent results we refer, e.g., to [1,5–11]. It also corresponds to the notion of shadowing in dynamical systems and computer sciences (see, e.g., [12,13]).

Following the terminology in [2,5,6,8–11] we say that recurrence (3) is $(\varepsilon_n)_{n\geqslant 0}$-stable provided, for every sequence $(x_n)_{n\geqslant 0}$ in X satisfying (2), there exists a sequence $(y_n)_{n\geqslant 0}$ in X such that (3) and (4) hold; otherwise, we say that the recurrence is $(\varepsilon_n)_{n\geqslant 0}$-nonstable.

In this paper we investigate stability of (3) in the case where condition (1) can be possibly not valid. First we show that in the situation

$$\lim_{n\to\infty}\frac{\varepsilon_n|a_{n+1}|}{\varepsilon_{n+1}}=1 \tag{5}$$

recurrence (3) is $(\varepsilon_n)_{n\geqslant 0}$-nonstable.

Let us start with the following simple observation.

Lemma 1. *If $(z_n)_{n\geqslant 0}$ and $(d_n)_{n\geqslant 0}$ are sequences in X and $z_{n+1}=a_nz_n+d_n$ for $n\geqslant 0$, then*

$$z_n=a_0\cdots a_{n-1}z_0+\sum_{k=1}^{n-1}a_k\cdots a_{n-1}d_{k-1}+d_{n-1},\quad n\geqslant 2. \tag{6}$$

Moreover, for each sequence $(x_n)_{n\geqslant 0}$ in X with $\|x_{n+1}-a_nx_n-d_n\|\leqslant\varepsilon_n$ for $n\geqslant 0$,

$$\|x_n-z_n\|\leqslant|a_0\cdots a_{n-1}|\|x_0-z_0\|+\sum_{i=1}^{n-1}|a_i\cdots a_{n-1}|\varepsilon_{i-1}+\varepsilon_{n-1},\quad n\geqslant 2. \tag{7}$$

Proof. By induction on n it is easy to show that (6) holds. Next, let $(x_n)_{n\geqslant 0}$ be a sequence in X such that $\|x_{n+1}-a_nx_n-d_n\|\leqslant\varepsilon_n$ for $n\geqslant 0$. Write $c_n:=x_{n+1}-a_nx_n-d_n$ for $n\geqslant 0$. Then, by (6) with z_n and d_n replaced by x_n and d_n+c_n, respectively, we obtain

$$x_n=a_0\cdots a_{n-1}x_0+\sum_{k=1}^{n-1}a_k\cdots a_{n-1}(d_{k-1}+c_{k-1})+d_{n-1}+c_{n-1}$$

for $n\geqslant 2$, which implies (7). □

Theorem 1. *Assume that condition* (5) *holds. Then there exists a sequence $(x_n)_{n\geqslant 0}$ in X satisfying* (2) *and such that, for every sequence $(y_n)_{n\geqslant 0}$ in X, given by* (3), *we have*

$$\sup_{n\in\mathbb{N}}\frac{\|x_n-y_n\|}{\varepsilon_{n-1}}=\infty. \tag{8}$$

Proof. Let $x\in X$, $0<\|x\|\leqslant 1$, and

$$c_n:=\varepsilon_n\frac{a_0\cdots a_n}{|a_0\cdots a_n|}x,\quad n\geqslant 0. \tag{9}$$

Take $x_0\in X$ and define $(x_n)_{n\geqslant 0}$ by the recurrence

$$x_{n+1}=a_nx_n+b_n+c_n,\quad n\geqslant 0. \tag{10}$$

Then

$$\|x_{n+1}-a_nx_n-b_n\|\leqslant\varepsilon_n,\quad n\geqslant 0,$$

whence relation (2) is satisfied.

Now let $(y_n)_{n\geqslant 0}$ be an arbitrary sequence satisfying recurrence (3). On account of Lemma 1 with $z_n:=x_n-y_n$ and $d_n:=c_n$, for each $n\in\mathbb{N}$ we get

$$\begin{aligned}x_n-y_n&=a_0\cdots a_{n-1}\left(x_0-y_0+\sum_{k=1}^{n}\frac{c_{k-1}}{a_0\cdots a_{k-1}}\right)\\&=a_0\cdots a_{n-1}(x_0-y_0+s_nx),\end{aligned} \tag{11}$$

where

$$s_n := \sum_{k=1}^{n} \frac{\varepsilon_{k-1}}{|a_0 \cdots a_{k-1}|}, \quad n \geqslant 1. \tag{12}$$

First consider the case where there is finite $s := \lim_{n\to\infty} s_n$.

If $y_0 \neq x_0 + sx$, then

$$\lim_{n\to\infty} \frac{\|x_n - y_n\|}{\varepsilon_{n-1}} = \lim_{n\to\infty} \frac{|a_0 \cdots a_{n-1}|}{\varepsilon_{n-1}} \|x_0 - y_0 + sx\| = \infty,$$

because $\lim_{n\to\infty} \frac{1}{\varepsilon_{n-1}} |a_0 \cdots a_{n-1}| = \infty$.

If $y_0 = x_0 + xs$, then, for each $n \in \mathbb{N}$,

$$\begin{aligned}
\frac{\|x_n - y_n\|}{\varepsilon_{n-1}} &= \frac{|a_0 \cdots a_{n-1}|}{\varepsilon_{n-1}} \big\|(s - s_n)x\big\| \\
&= \frac{|a_0 \cdots a_{n-1}|}{\varepsilon_{n-1}} (s - s_n)\|x\| = \frac{s - s_n}{s_n - s_{n-1}} \|x\| \\
&= \frac{\|x\|}{\frac{s - s_{n-1}}{s - s_n} - 1}
\end{aligned}$$

and next, according to the well-known Stolz–Cesaro lemma (case $\frac{0}{0}$),

$$\begin{aligned}
\lim_{n\to\infty} \frac{s - s_{n-1}}{s - s_n} &= \lim_{n\to\infty} \frac{(s - s_n) - (s - s_{n-1})}{(s - s_{n+1}) - (s - s_n)} = \lim_{n\to\infty} \frac{s_{n-1} - s_n}{s_n - s_{n+1}} \\
&= \lim_{n\to\infty} \frac{s_n - s_{n-1}}{s_{n+1} - s_n} = \lim_{n\to\infty} \frac{\varepsilon_{n-1}|a_n|}{\varepsilon_n} = 1,
\end{aligned}$$

whence we have (8).

Now, suppose that $\lim_{n\to\infty} s_n = \infty$. If

$$\limsup_{n\to\infty} \frac{|a_0 \cdots a_{n-1}|}{\varepsilon_{n-1}} \neq 0,$$

then (8) holds (in view of (11)), because $\lim_{n\to\infty} \|x_0 - y_0 + s_n x\| = \infty$.

So, it remains to consider the case

$$\lim_{n\to\infty} \frac{|a_0 \cdots a_{n-1}|}{\varepsilon_{n-1}} = 0.$$

Clearly, from (11) it follows that, for every $n \in \mathbb{N}$,

$$\frac{x_n - y_n}{\varepsilon_{n-1}} = \frac{a_0 \cdots a_{n-1}}{\varepsilon_{n-1}} (x_0 - y_0) + \frac{a_0 \cdots a_{n-1}}{\varepsilon_{n-1}} s_n x. \tag{13}$$

Write

$$z_n := \frac{a_0 \cdots a_{n-1}}{\varepsilon_{n-1}} s_n x, \quad n \in \mathbb{N}.$$

Then, in view of the monotonicity and unboundedness of $(s_n)_{n\geqslant 0}$, by the Stolz–Cesaro lemma we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{\|x\|}{\|z_n\|} &= \lim_{n\to\infty} \frac{\frac{\varepsilon_n}{|a_0 \cdots a_n|} - \frac{\varepsilon_{n-1}}{|a_0 \cdots a_{n-1}|}}{s_{n+1} - s_n} \\
&= \lim_{n\to\infty} \left(1 - \frac{|a_n|\varepsilon_{n-1}}{\varepsilon_n}\right) = 0, \quad n \in \mathbb{N},
\end{aligned}$$

and therefore $\lim_{n\to\infty} \|z_n\| = \infty$. Since $\lim_{n\to\infty} \frac{a_0 \cdots a_{n-1}}{\varepsilon_{n-1}} (x_0 - y_0) = 0$, in view of (13) this yields

$$\lim_{n\to\infty} \frac{\|x_n - y_n\|}{\varepsilon_{n-1}} = \infty. \qquad \square$$

Remark 1. It is easily seen in the proof of Theorem 1 that changing the vector x we obtain a large class of examples of sequences $(x_n)_{n\geqslant 0}$ in X satisfying (2) and (8) with every sequence $(y_n)_{n\geqslant 0}$ in X, given by (3). Clearly, the cardinality of that class is not less than the cardinality of the space X.

In connection with Theorem 1 the following question seems to be very natural.

Question 1. Can we replace condition (5) in Theorem 1 by $\liminf_{n\to\infty} \frac{|a_{n+1}|\varepsilon_n}{\varepsilon_{n+1}} = 1$ or $\limsup_{n\to\infty} \frac{|a_{n+1}|\varepsilon_n}{\varepsilon_{n+1}} = 1$?

The next four examples show that probably there is no simple answer to that question.

Example 1. Let ε be a positive real number, $\varepsilon_n = \varepsilon$, $a_{2n} = 1$, and $a_{2n+1} = 2$ for $n \geqslant 0$. Then $\liminf_{n\to\infty} \frac{|a_{n+1}|\varepsilon_n}{\varepsilon_{n+1}} = 1$ and recurrence (3) is $(\varepsilon_n)_{n\geqslant 0}$-stable.

Proof. The $(\varepsilon_n)_{n\geqslant 0}$-stability of (3) can be easily derived from [15, Theorem 2.1]. However, for the convenience of the reader we present a short direct proof of it.

Let $(x_n)_{n\geqslant 0}$ a sequence in X such that (2) holds. Write $c_n := x_{n+1} - a_n x_n - b_n$, $n \geqslant 0$. It is easily seen that there exists

$$s := \sum_{n=1}^{\infty} \frac{c_{n-1}}{a_0 \cdots a_{n-1}}.$$

Let $y_{n+1} = a_n y_n + b_n$ for $n \geqslant 0$ and $y_0 := x_0 + s$. Then (cf. (11))

$$\begin{aligned}\|x_n - y_n\| &= |a_0 \cdots a_{n-1}| \cdot \left\| -s + \sum_{k=1}^{n} \frac{c_{k-1}}{a_0 \cdots a_{k-1}} \right\| \\ &= |a_0 \cdots a_{n-1}| \cdot \left\| \sum_{k=n}^{\infty} \frac{c_k}{a_0 \cdots a_k} \right\| \leqslant M\varepsilon,\end{aligned}$$

where

$$M = \sum_{k=0}^{\infty} \frac{1}{|a_n \cdots a_{n+k}|} = \begin{cases} 3, & \text{if } n \text{ is even;} \\ 2, & \text{if } n \text{ is odd.} \end{cases} \qquad \square$$

Example 2. Let ε be a positive real number, $\varepsilon_n = \varepsilon$, $r \in \mathbb{N}$, $r > 1$, $a_{r^n} = 2$ for $n \geqslant 0$, and $a_k = 1$ for $k \notin \{r^n\colon\ n \geqslant 0\}$. Then $\liminf_{n\to\infty} \frac{|a_{n+1}|\varepsilon_n}{\varepsilon_{n+1}} = 1$ and recurrence (3) is $(\varepsilon_n)_{n\geqslant 0}$-nonstable.

Proof. Let $(x_n)_{n\geqslant 0}$ be defined by (10), with $(c_n)_{n\geqslant 0}$ given by (9), and $(y_n)_{n\geqslant 0}$ be an arbitrary sequence satisfying recurrence (3). Then, analogously as in the proof of Theorem 1, we get (11). Since

$$\sum_{n=0}^{\infty} \frac{1}{|a_0 \cdots a_n|} = \infty, \qquad \lim_{n\to\infty} |a_0 \cdots a_{n-1}| = \infty,$$

this yields

$$\lim_{n\to\infty} \|x_n - y_n\| = \infty. \qquad \square$$

Example 3. Let ε be a positive real number, $\varepsilon_n = \varepsilon$, $a_{2n} = \frac{1}{2}$ and $a_{2n+1} = 1$ for $n \geqslant 0$. Then $\limsup_{n\to\infty} \frac{|a_{n+1}|\varepsilon_n}{\varepsilon_{n+1}} = 1$ and recurrence (3) is $(\varepsilon_n)_{n\geqslant 0}$-stable.

Proof. Let $(x_n)_{n\geqslant 0}$ be an arbitrary sequence satisfying (2), $(y_n)_{n\geqslant 0}$ be a sequence satisfying (3) with $y_0 := x_0$, and $c_n := x_{n+1} - a_n x_n - b_n$ for $n \geqslant 0$. Then $\|c_n\| \leqslant \varepsilon$ and, by Lemma 1, (6) holds with $z_n := x_n - y_n$ and $d_n := c_n$ for $n \geqslant 0$, whence

$$\begin{aligned}\|x_n - y_n\| &\leqslant |a_0 \cdots a_{n-1}| \cdot \left\| \sum_{k=1}^{n} \frac{c_{k-1}}{a_0 \cdots a_{k-1}} \right\| \\ &\leqslant \varepsilon |a_0 \cdots a_{n-1}| \sum_{k=1}^{n} \frac{1}{|a_0 \cdots a_{k-1}|}, \quad n \geqslant 1.\end{aligned}$$

Hence, for each $n \geqslant 1$,

$$\begin{aligned}\|x_{2n} - y_{2n}\| &\leqslant \varepsilon \cdot \frac{1}{2^n}\left(2 + 2 + 2^2 + 2^2 + \cdots + 2^n + 2^n\right) \\ &= 4\varepsilon \cdot \frac{2^n - 1}{2^n} < 4\varepsilon,\end{aligned}$$

$$\begin{aligned}\|x_{2n+1} - y_{2n+1}\| &\leqslant \varepsilon \cdot \frac{1}{2^{n+1}}\bigl(2+2+2^2+2^2+\cdots+2^n+2^n+2^{n+1}\bigr)\\ &= 4\varepsilon \cdot \frac{2^n-1}{2^{n+1}} + \varepsilon < 3\varepsilon. \qquad \square\end{aligned}$$

Example 4. Let ε be a positive real number, $\varepsilon_n = \varepsilon$ for $n \geqslant 0$, $I := \{2^k - 2\colon\ k \in \mathbb{N}\}$, $a_n = \frac{1}{2}$ for $n \in I$, and $a_n = 1$ for $n \notin I$. Then $\limsup_{n\to\infty} \frac{|a_{n+1}|\varepsilon_n}{\varepsilon_{n+1}} = 1$ and recurrence (3) is $(\varepsilon_n)_{n\geqslant 0}$-nonstable.

Proof. Let $(x_n)_{n\geqslant 0}$ be defined by (10) with $(c_n)_{n\geqslant 0}$ given by (9) and $(y_n)_{n\geqslant 0}$ be an arbitrary sequence satisfying recurrence (3). Then, by Lemma 1 (with $z_n := x_n - y_n$ and $d_n := c_n$ for $n \geqslant 0$), for every $n > 1$,

$$\begin{aligned}x_{2^n-2} - y_{2^n-2} &= a_0\cdots a_{2^n-3}\left(x_0 - y_0 + \varepsilon x \sum_{k=1}^{2^n-2} \frac{1}{|a_0\cdots a_{k-1}|}\right)\\ &= \frac{1}{2^{n-1}}\Bigl(x_0 - y_0 + \varepsilon x\bigl(2+2+\bigl(2^2+2^2+2^2+2^2\bigr)+\cdots+\bigl(\underbrace{2^{n-1}+\cdots+2^{n-1}}_{2^{n-1}\text{ terms}}\bigr)\bigr)\Bigr)\\ &= \frac{1}{2^{n-1}}\bigl(x_0 - y_0 + \varepsilon x\bigl(2^2+2^4+\cdots+2^{2n-2}\bigr)\bigr)\\ &= \frac{1}{2^{n-1}}\left(x_0 - y_0 + 2^2\cdot\frac{2^{2n-2}-1}{3}\cdot\varepsilon x\right).\end{aligned}$$

Hence

$$\lim_{n\to\infty} \|x_{2^n-2} - y_{2^n-2}\| = \infty. \qquad \square$$

As we have already mentioned in the proof of Example 1, some stability results for recurrence (3), in the case where (1) does not hold, can be obtained from [15]. Namely, from [15, Theorem 2.1] the following proposition can be easily derived.

Proposition 1. *Let $(x_n)_{n\geqslant 0}$ be a sequence in X, (2) hold, and*

$$A_n := \sum_{k=0}^{\infty} \frac{\varepsilon_{n+k}}{|a_n\cdots a_{n+k}|} < \infty, \quad n \geqslant 0. \tag{14}$$

Then there is a unique sequence $(y_n)_{n\geqslant 0}$ in X, satisfying recurrence (3)*, with*

$$\|x_n - y_n\| \leqslant A_n, \quad n \geqslant 0. \tag{15}$$

The next proposition generalizes Proposition 1.

Proposition 2. *Assume that $(x_n)_{n\geqslant 0}$ is a sequence in X,* (2) *holds, there exists*

$$\rho := \sum_{k=0}^{\infty} \frac{1}{a_0\cdots a_k}(x_{k+1} - a_k x_k - b_k), \tag{16}$$

and $(y_n)_{n\geqslant 0}$ is a sequence in X given by (3) *with $y_0 = x_0 + \rho$. Then*

$$\|x_n - y_n\| \leqslant \sum_{k=0}^{\infty} \frac{\varepsilon_{n+k}}{|a_n\cdots a_{n+k}|} =: A_n, \quad n \geqslant 0. \tag{17}$$

Moreover, if $A_n < \infty$ for each $n \geqslant 0$ and the sequence

$$\left(\frac{A_n}{\varepsilon_{n-1}}\right)_{n\in\mathbb{N}}$$

is bounded, then $(y_n)_{n\geqslant 0}$ is a unique sequence in X satisfying (3) *and* (4)*.*

Proof. As in the proof of Example 1, for each $n \in \mathbb{N}$ we get

$$\|x_n - y_n\| = |a_0 \cdots a_{n-1}| \cdot \left\| \sum_{k=0}^{\infty} \frac{1}{a_0 \cdots a_{n+k}} c_{n+k} \right\| \leqslant A_n.$$

So, it remains to show the statement concerning uniqueness.

To this end suppose $(w_n)_{n \geqslant 0}$ is a sequence in X, $w_{n+1} = a_n w_n + b_n$ for $n \geqslant 0$ and $w_0 \neq x_0 + \rho$. Then, by (6) (with $z_n := x_n - w_n$ and $d_n := c_n$),

$$\limsup_{n\to\infty} \frac{\|x_n - w_n\|}{\varepsilon_{n-1}} = \limsup_{n\to\infty} \frac{|a_0 \cdots a_{n-1}|}{\varepsilon_{n-1}} \|x_0 - w_0 + \rho\| = \infty,$$

because we assume that $A_0 < \infty$. $\square$

From Lemma 1 (condition (6) with $z_0 := x_0$) we get at once the next proposition, which is somewhat complementary to Proposition 1.

Proposition 3. *Let $(x_n)_{n \geqslant 0}$ and $(y_n)_{n \geqslant 0}$ be sequences in X such that* (2) *and* (3) *hold and $y_0 = x_0$. Then*

$$\frac{\|x_n - y_n\|}{\varepsilon_{n-1}} \leqslant 1 + \frac{1}{\varepsilon_{n-1}} \sum_{i=1}^{n-1} |a_i \cdots a_{n-1}| \varepsilon_{i-1}, \quad n \geqslant 1. \tag{18}$$

Remark 2. Let $(x_n)_{n \geqslant 0}$ be a sequence in X such that (2) holds and let the sequences

$$\left(\frac{1}{\varepsilon_n} \sum_{i=1}^{n} |a_i \cdots a_n| \varepsilon_{i-1} \right)_{n \in \mathbb{N}}, \qquad \left(\frac{|a_0 \cdots a_n|}{\varepsilon_n} \right)_{n \in \mathbb{N}}$$

be bounded. Then condition (7) (with $z_0 := y_0$) implies (4) for every sequence $(y_n)_{n \geqslant 0}$ in X, satisfying recurrence (3). So, in such situation, there is no uniqueness of the sequence $(y_n)_{n \geqslant 0}$ satisfying (3) and (4).

We finish this paper with two examples of situations, where Propositions 1 and 3 can be applied.

Example 5. Let $(x_n)_{n \geqslant 0}$ be a sequence in X such that (2) holds, $C > 0$, $(\delta_n)_{n \geqslant 0}$ be a sequence in $(0, \infty)$, $\delta_{n+k} \leqslant C\delta_{n-1}\delta_k$ for $n, k \in \mathbb{N}$ and

$$\Delta := \sum_{k=0}^{\infty} \delta_k < \infty. \tag{19}$$

Then the following two statements are valid.

(α) If

$$S_1 := \limsup_{n\to\infty} \frac{|a_0 \cdots a_n| \delta_n}{\varepsilon_n} < \infty, \qquad I_1 := \liminf_{n\to\infty} \frac{|a_0 \cdots a_n| \delta_n}{\varepsilon_n} > 0,$$

then there exists a unique sequence $(y_n)_{n \geqslant 0}$ in X, satisfying recurrence (3), with

$$\sup_{n \in \mathbb{N}} \frac{\|x_n - y_n\|}{\varepsilon_{n-1}} < \infty;$$

moreover, $y_0 = x_0 + \rho$ (where ρ is defined by (16)) and

$$\limsup_{n\to\infty} \frac{\|x_n - y_n\|}{\varepsilon_{n-1}} \leqslant \frac{CS_1}{I_1} \Delta. \tag{20}$$

(β) If

$$S_2 := \limsup_{n\to\infty} \frac{|a_0 \cdots a_n|}{\delta_n \varepsilon_n} < \infty, \qquad I_2 := \liminf_{n\to\infty} \frac{|a_0 \cdots a_n|}{\delta_n \varepsilon_n} > 0,$$

then there is a sequence $(y_n)_{n \geqslant 0}$ in X satisfying (3) with

$$\limsup_{n\to\infty} \frac{\|x_n - y_n\|}{\varepsilon_{n-1}} \leqslant 1 + \frac{CS_2}{I_2} \Delta. \tag{21}$$

Proof. (α) Take $P_1 > S_1$ and $0 < p_1 < I_1$. There exists $n_1 \in \mathbb{N}$ such that

$$p_1 \leqslant \frac{|a_0 \cdots a_n|\delta_n}{\varepsilon_n} \leqslant P_1, \quad n \geqslant n_1,$$

whence

$$\frac{\varepsilon_{n+k}}{|a_n \cdots a_{n+k}|\varepsilon_{n-1}\delta_k} \leqslant \frac{|a_0 \cdots a_{n-1}|C\varepsilon_{n+k}\delta_{n-1}}{|a_0 \cdots a_{n+k}|\varepsilon_{n-1}\delta_{n+k}} \leqslant \frac{CP_1}{p_1}$$

for $n > n_1$, $k \geqslant 0$. Hence, by (19), (14) holds and consequently Proposition 1 yields the statement.

(β) Let $P_2 > S_2$ and $0 < p_2 < I_2$. There is $n_2 \in \mathbb{N}$ such that

$$p_2 \leqslant \frac{|a_0 \cdots a_n|}{\delta_n \varepsilon_n} \leqslant P_2, \quad n \geqslant n_2.$$

Hence

$$\frac{|a_n \cdots a_{n+k}|\varepsilon_{n-1}}{\delta_k \varepsilon_{n+k}} \leqslant \frac{|a_0 \cdots a_{n+k}|C\delta_{n-1}\varepsilon_{n-1}}{|a_0 \cdots a_{n-1}|\delta_{n+k}\varepsilon_{n+k}} \leqslant \frac{CP_2}{p_2}, \tag{22}$$

for $n > n_2$, $k \geqslant 0$. Let $(y_n)_{n\geqslant 0}$ be a sequence in X satisfying recurrence (3) with $y_0 = x_0$. Since, according to Proposition 3, (18) is valid, (22) implies the statement. □

Remark 3. It is easily seen that the stability of the recurrences from Examples 1 and 3 follows from Example 5 with $\delta_n := (\sqrt{2})^{-n}$.

Example 6. Suppose there is a sequence of positive real numbers $(\lambda_n)_{n\geqslant 0}$ with

$$\frac{\varepsilon_n |a_{n+1}|}{\varepsilon_{n+1}} \geqslant \frac{\lambda_n}{\lambda_{n+1}}(\lambda_{n+1} + 1), \quad n \geqslant 0. \tag{23}$$

Then

$$A_0 := \sum_{k=0}^{\infty} \frac{\varepsilon_k}{|a_0 \cdots a_k|} \leqslant \frac{\varepsilon_0}{|a_0|}\left(1 + \frac{1}{\lambda_0}\right),$$

$$A_n := \sum_{k=0}^{\infty} \frac{\varepsilon_{n+k}}{|a_n \cdots a_{n+k}|} \leqslant \frac{1}{\lambda_{n-1}(\lambda_n + 1)} \cdot \varepsilon_{n-1}, \quad n \geqslant 1,$$

and, for each sequence $(x_n)_{n\geqslant 0}$ in X, satisfying (2), there exists

$$\rho := \sum_{k=0}^{\infty} \frac{1}{a_0 \cdots a_k}(x_{k+1} - a_k x_k - b_k).$$

Proof. It is enough to notice that, for every $k \geqslant 0$, $n > 0$,

$$\begin{aligned}\frac{\varepsilon_n}{|a_0 \cdots a_n|} &\leqslant \frac{\varepsilon_0}{|a_0|\lambda_0} \cdot \frac{\lambda_n}{(\lambda_1 + 1)\cdots(\lambda_n + 1)} \\ &= \frac{\varepsilon_0}{|a_0|\lambda_0} \cdot \left(\frac{1}{(\lambda_1 + 1)\cdots(\lambda_{n-1} + 1)} - \frac{1}{(\lambda_1 + 1)\cdots(\lambda_n + 1)}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\varepsilon_{n+k}}{|a_n \cdots a_{n+k}|} &\leqslant \frac{\varepsilon_{n-1}}{\lambda_{n-1}} \cdot \frac{\lambda_{n+k}}{(\lambda_n + 1)\cdots(\lambda_{n+k} + 1)} \\ &= \frac{\varepsilon_{n-1}}{\lambda_{n-1}}\left(\frac{1}{(\lambda_n + 1)\cdots(\lambda_{n+k-1} + 1)} - \frac{1}{(\lambda_n + 1)\cdots(\lambda_{n+k} + 1)}\right).\end{aligned}$$

Consequently, the limit ρ exists. □

References

[1] R.P. Agarwal, B. Xu, W. Zhang, Stability of functional equations in single variable, J. Math. Anal. Appl. 288 (2003) 852–869.
[2] J. Brzdęk, D. Popa, B. Xu, Note on nonstability of the linear recurrence, Abh. Math. Sem. Univ. Hamburg 76 (2006) 183–189.
[3] J. Brzdęk, D. Popa, B. Xu, The Hyers–Ulam stability of nonlinear recurrences, J. Math. Anal. Appl. 335 (2007) 443–449.
[4] J. Brzdęk, D. Popa, B. Xu, The Hyers–Ulam stability of linear equations of higher orders, Acta Math. Hungar. 120 (2008) 1–8.
[5] S. Czerwik, Functional Equations and Inequalities in Several Variables, World Scientific, London, 2002.
[6] G.L. Forti, Hyers–Ulam stability of functional equations in several variables, Aequationes Math. 50 (1995) 143–190.
[7] G.L. Forti, Comments on the core of the direct method for proving Hyers–Ulam stability of functional equations, J. Math. Anal. Appl. 295 (2004) 127–133.
[8] D.H. Hyers, G. Isac, Th.M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Boston–Basel–Berlin, 1998.
[9] S.-M. Jung, Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, FL, 2001.
[10] L. Maligranda, A result of Tosio Aoki about a generalization of Hyers–Ulam stability of additive functions – a question of priority, Aequationes Math. 75 (2008) 289–296.
[11] Z. Moszner, On the stability of functional equations, Aequationes Math. 77 (2009) 33–88.
[12] K. Palmer, Shadowing in Dynamical Systems, Kluwer Academic Press, 2000.
[13] S. Pilyugin, Shadowing in Dynamical Systems, Lecture Notes in Math., vol. 1706, Springer-Verlag, 1999.
[14] D. Popa, Hyers–Ulam–Rassias stability of a linear recurrence, J. Math. Anal. Appl. 309 (2005) 591–597.
[15] T. Trif, On the stability of a general gamma-type functional equation, Publ. Math. Debrecen 60 (2002) 47–61.